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Surfaces of Constant Mean Curvature.

By L. P. Eisenhart.

Cosserat has established the following theorem :*

Let ϕ be any function of two variables u and v; if z denotes any solution of the equation

$$\frac{\partial^2 z}{\partial u\,\partial v} - \frac{1}{2}\frac{\dfrac{\partial^2\phi}{\partial u\,\partial v}}{\dfrac{\partial\phi}{\partial u}}\frac{\partial z}{\partial u} - \frac{1}{2}\frac{\dfrac{\partial^2\phi}{\partial u\,\partial v}}{\dfrac{\partial\phi}{\partial v}}\frac{\partial z}{\partial v} = 0, \tag{1}$$

the formulæ

$$x + iy = \phi,\quad x - iy = -\int \frac{\left(\dfrac{\partial z}{\partial u}\right)^2}{\dfrac{\partial\phi}{\partial u}}\,du + \frac{\left(\dfrac{\partial z}{\partial v}\right)^2}{\dfrac{\partial\phi}{\partial v}}\,dv, \qquad z = z, \tag{2}$$

give the cartesian coordinates x, y, z of a surface for which the parametric lines are of length zero; and the square of the linear element is

$$ds^2 = -\frac{\left(\dfrac{\partial z}{\partial u}\dfrac{\partial\phi}{\partial v} - \dfrac{\partial z}{\partial v}\dfrac{\partial\phi}{\partial u}\right)^2}{\dfrac{\partial\phi}{\partial u}\dfrac{\partial\phi}{\partial v}}\,du\,dv. \tag{3}$$

This method for the determination of a surface lends itself particularly to surfaces of constant mean curvature. We find that for this class of surfaces the function ϕ satisfies a partial differential equation of the fourth order, and that when a particular integral is found, the further determination of the corresponding surface requires quadratures only. Several particular integrals can be found; in some of these cases the surface is imaginary.

* Comptes Rendus, 125, pp. 159–162.

From (3) we have

$$E=\Sigma\left(\frac{\partial x}{\partial u}\right)^2=0, \quad F=\Sigma\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v}=-\tfrac{1}{2}\,\frac{\left(\dfrac{\partial z}{\partial u}\,\dfrac{\partial \phi}{\partial v}-\dfrac{\partial z}{\partial v}\,\dfrac{\partial \phi}{\partial u}\right)^2}{\dfrac{\partial \phi}{\partial u}\,\dfrac{\partial \phi}{\partial v}},$$

$$G=\Sigma\left(\frac{\partial x}{\partial v}\right)^2=0. \tag{4}$$

Differentiating the first of these expressions with respect to u, we have

$$\Sigma\frac{\partial x}{\partial u}\,\frac{\partial^2 x}{\partial u^2}=0,$$

Associate with this the identity

$$\Sigma X\frac{\partial x}{\partial u}=0,$$

where X, Y, Z denote the direction cosines of the normal to the surface, then we have the relations

$$\frac{\dfrac{\partial x}{\partial u}}{Y\dfrac{\partial^2 z}{\partial u^2}-Z\dfrac{\partial^2 y}{\partial u^2}}=\frac{\dfrac{\partial y}{\partial u}}{Z\dfrac{\partial^2 x}{\partial u^2}-X\dfrac{\partial^2 z}{\partial u^2}}=\frac{\dfrac{\partial z}{\partial u}}{X\dfrac{\partial^2 y}{\partial u^2}-Y\dfrac{\partial^2 z}{\partial u^2}}.$$

In consequence of (4) we have

$$\Sigma\left(Y\frac{\partial^2 z}{\partial u^2}-Z\frac{\partial^2 y}{\partial u^2}\right)^2=\Sigma X^2.\Sigma\left(\frac{\partial^2 x}{\partial u^2}\right)^2-\left(\Sigma X\frac{\partial^2 x}{\partial u^2}\right)^2=0,$$

and hence if we adopt the notation

$$D=\Sigma X\frac{\partial^2 x}{\partial u^2}, \quad D'=\Sigma X\frac{\partial^2 x}{\partial u\,\partial v}, \quad D''=\Sigma X\frac{\partial^2 x}{\partial v^2},$$

the above identity gives

$$D^2=\Sigma\left(\frac{\partial^2 x}{\partial u^2}\right)^2; \tag{5}$$

in like manner it can be shown that

$$D'^2=\Sigma\left(\frac{\partial^2 x}{\partial u\,\partial v}\right)^2, \quad D''^2=\Sigma\left(\frac{\partial^2 x}{\partial v^2}\right)^2. \tag{6}$$

Substituting for x, y, z their expressions from (2) and making use of (1), we get

$$D^2 = \frac{\left(\frac{\partial^2\phi}{\partial u^2}\frac{\partial z}{\partial u} - \frac{\partial\phi}{\partial u}\frac{\partial^2 z}{\partial u^2}\right)^2}{\left(\frac{\partial\phi}{\partial u}\right)^2}, \quad D''^2 = \frac{\left(\frac{\partial^2\phi}{\partial v^2}\frac{\partial z}{\partial v} - \frac{\partial\phi}{\partial v}\frac{\partial^2 z}{\partial v^2}\right)^2}{\left(\frac{\partial\phi}{\partial v}\right)^2},$$

$$D'^2 = \tfrac{1}{4}\left(\frac{\partial z}{\partial u}\frac{\partial\phi}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial\phi}{\partial u}\right)^2 \frac{\left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2}{\left(\frac{\partial\phi}{\partial u}\right)^2\left(\frac{\partial\phi}{\partial v}\right)^2}, \tag{7}$$

The expression for the mean curvature takes the simple form*

$$\tfrac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) = -\frac{D'}{F}, \tag{8}$$

where ρ_1 and ρ_2 are the principal radii of curvature; hence for surfaces of constant mean curvature $\varkappa$, we have

$$\frac{D'}{F} = -\varkappa.$$

Substituting D' and F their expressions from (7) and (4), we have

$$\frac{\partial^2\phi}{\partial u\,\partial v} = \pm\varkappa\left(\frac{\partial z}{\partial u}\frac{\partial\phi}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial\phi}{\partial u}\right).$$

Since the sign of the mean curvature of a surface is determined by the assigned positive direction of the normal, it is evident that there is no loss of generality if we write the above expression as follows:

$$\frac{\partial^2\phi}{\partial u\,\partial v} = \varkappa\left(\frac{\partial z}{\partial u}\frac{\partial\phi}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial\phi}{\partial u}\right). \tag{9}$$

By retracing the steps in the above development, we find that the converse of this result is true. Hence, the necessary and sufficient condition that formulæ (2) define a surface of constant mean curvature $\varkappa$ is that ϕ and z satisfy equations (1) and (9) simultaneously.

When $\varkappa = 0$, that is, when the surface is minimal, equations (9) and (1) reduce to

$$\frac{\partial^2\phi}{\partial u\,\partial v} = 0, \quad \frac{\partial^2 z}{\partial u\,\partial v} = 0,$$

* Bianchi, Lezioni, p. 104.

hence

$$\phi = U_1 + V_1, \quad z = U_2 + V_2,$$

where U_1 and U_2 are functions of u alone, V_1 and V_2 are functions of v alone. Substituting these expressions for ϕ and z in (2), we find the well-known property of minimal surfaces, namely, that their cartesian coordinates are expressible as a sum of two functions, each of one of the parameters of its lines of length zero. That this is a characteristic property can be seen from (9). For, in order that $\frac{\partial^2\phi}{\partial u\,\partial v}$ be zero, it is necessary that $\varkappa$ be zero; otherwise

$$\frac{\partial z}{\partial u}\,\frac{\partial \phi}{\partial v} - \frac{\partial z}{\partial v}\,\frac{\partial \phi}{\partial u} = 0,$$

that is, z is expressible as a function of ϕ and likewise x and y, and consequently the formulæ (2) would define a curve and not a surface. We will not discuss any further the case where $\varkappa = 0$, but in what follows it will be understood that $\varkappa \neq 0$.

From (9) we see that if ϕ is a function of u alone or a function of v alone, the above equation holds, and consequently the surface reduces to a curve. Hence ϕ is a function of both u and v.

Eliminating z between equations (1) and (9), we find that for equations (2) to define a surface of constant mean curvature $\varkappa$, the function ϕ must satisfy the following partial differential equation of the fourth order:

$$\frac{\partial^4\phi}{\partial u^2\partial v^2} = \left(\frac{\frac{\partial^2\phi}{\partial u^2}}{\frac{\partial\phi}{\partial u}} + \frac{\frac{\partial^2\phi}{\partial u\,\partial v}}{\frac{\partial\phi}{\partial v}}\right)\frac{\partial^3\phi}{\partial u\,\partial v^2} + \left(\frac{\frac{\partial^2\phi}{\partial u\,\partial v}}{\frac{\partial\phi}{\partial u}} + \frac{\frac{\partial^2\phi}{\partial v^2}}{\frac{\partial\phi}{\partial v}}\right)\frac{\partial^3\phi}{\partial u^2\,\partial v}$$

$$- \frac{\frac{\partial^2\phi}{\partial u^2}\,\frac{\partial^2\phi}{\partial u\,\partial v}\,\frac{\partial^2\phi}{\partial v^2}}{\frac{\partial\phi}{\partial u}\,\frac{\partial\phi}{\partial v}} - \left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2\left(\frac{\frac{\partial^2\phi}{\partial u^2}}{\left(\frac{\partial\phi}{\partial u}\right)^2} + \frac{\frac{\partial^2\phi}{\partial v^2}}{\left(\frac{\partial\phi}{\partial v}\right)^2}\right). \quad (10)$$

It is of interest to note that this equation does not involve $\varkappa$, and hence the general solution leads to surfaces of any constant mean curvature whatever.

Solve equation (9) for $\frac{\partial z}{\partial v}$ and substitute its expression in (1); then, by a

quadrature, we find

$$\frac{\partial z}{\partial u}=\frac{1}{2\varkappa}\frac{\partial\phi}{\partial u}\left(-\int\frac{\left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2}{\left(\frac{\partial\phi}{\partial u}\right)^2\frac{\partial\phi}{\partial v}}dv+U\right),\tag{11}$$

where U is a function of u alone, whose form is perfectly determinate for a value of ϕ, as we shall see in a moment. In a similar way we find

$$\frac{\partial z}{\partial v}=\frac{1}{2\varkappa}\frac{\partial\phi}{\partial v}\left(\int\frac{\left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2}{\frac{\partial\phi}{\partial u}\left(\frac{\partial\phi}{\partial v}\right)^2}du+V\right),\tag{12}$$

where V is a determinate function of v alone. Since $\frac{\partial^2\phi}{\partial u\,\partial v}\neq 0$, the condition of integrability of the expressions (11) and (12) reduces to

$$\int\frac{\left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2}{\frac{\partial\phi}{\partial u}\left(\frac{\partial\phi}{\partial v}\right)^2}du+\int\frac{\left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2}{\left(\frac{\partial\phi}{\partial u}\right)^2\frac{\partial\phi}{\partial v}}dv+2\frac{\frac{\partial^2\phi}{\partial u\,\partial v}}{\frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial v}}+V-U=0.\tag{13}$$

If this equation be differentiated with respect to u and v, the resulting equation can be brought to the form (10). Hence, for every particular integral ϕ of (10), the sum of the first three terms of (13) reduces to the sum of a function of u alone and a function of v alone. Thus, given an integral ϕ_1, we find that the sum of the first three terms reduces to U_1+V_1, where U_1 and V_1 are readily found. Then from (13) we have

$$U=U_1+c,\quad V=-V_1+c,$$

and, consequently, U and V are known. Therefore, having found an integral of (10) and the corresponding function U and V by means of (13), we get the z coordinate of the corresponding surface of mean curvature $\varkappa$ by the quadrature

$$z=\frac{1}{2\varkappa}\int\left[\frac{\partial\phi}{\partial u}\left(-\int\frac{\left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2}{\left(\frac{\partial\phi}{\partial u}\right)^2\frac{\partial\phi}{\partial v}}dv+U\right)du\right.$$
$$\left.+\frac{\partial\phi}{\partial v}\left(\int\frac{\left(\frac{\partial^2\phi}{\partial u\,\partial v}\right)^2}{\frac{\partial\phi}{\partial u}\left(\frac{\partial\phi}{\partial v}\right)^2}du+V\right)dv\right].\tag{14}$$

From the form of equation (10) we see that an integral is determined only to within a constant factor, and we shall find that this factor fixes the magnitude of the mean curvature. Consider an integral ϕ of (10) which does not involve a constant factor, and in (2) and (14) replace ϕ by $c\phi$, where c is a constant. In place of U and V we must put U/c and V/c in consequence of (13). From this we see that the quantity under the integral sign does not vary with c. By means of (14) the expression for $x - iy$ can be put in the form $\frac{1}{c\varkappa^2}\Phi$, where Φ is a function independent of c and $\varkappa$. Consider now two surfaces, of coordinates x_1, y_1, z_1 and x_2, y_2, z_2 corresponding to the same function ϕ and the values c_1 and c_2 of the constant c. It is evident that the mean curvatures of the two surfaces are unequal; call them $\varkappa_1$ and $\varkappa_2$. Then we get from (2)

$$\frac{x_1 + iy_1}{c_1} = \frac{x_2 + iy_2}{c_2}, \quad c_1\varkappa_1^2(x_1 - iy_1) = c_2\varkappa_2^2(x_2 - iy_2),$$

from which it follows that

$$c_1\varkappa_1 = c_2\varkappa_2.$$

Hence, by choosing a suitable unit we can put $c = \frac{1}{\varkappa}$. We have then the following theorem:

Given an integral ϕ of equation (10) *which does not involve a constant factor; the formulæ*

$$\varkappa + iy = \phi/\varkappa, \quad x - iy = -\varkappa\int \frac{\left(\frac{\partial z}{\partial u}\right)^2}{\frac{\partial \phi}{\partial u}}du + \frac{\left(\frac{\partial z}{\partial v}\right)^2}{\frac{\partial \phi}{\partial v}}dv, \tag{15}$$

and (14) *define a surface of constant mean curvature* $\varkappa$.

However, all surfaces defined by these formulæ and corresponding to the same integral ϕ are homothetic to one another, and consequently, for the further discussion, there will be no lack of generality if we put $\varkappa = 1$ in (15).

Suppose we put $u = U_2$ and $v = V_2$, where U_2 and V_2 are any functions whatever of new parameters u_1 and v_1 respectively, defined in this manner, and denote by ϕ_1 the result of replacing u and v in ϕ by u_1 and v_1 respectively. It is evident that if ϕ is an integral of equation (10), ϕ_1 is an integral of the equation

obtained from (10) by replacing u and v by u_1 and v_1; we shall refer to this equation as (10′). From the above we have

$$\frac{\partial\phi_1}{\partial u_1}=\frac{\partial\phi_1}{\partial u}U_2',\quad \frac{\partial\phi_1}{\partial v_1}=\frac{\partial\phi_1}{\partial v}V_2',\ \ldots\ldots,$$

where the primes denote differentiation. If these values for $\frac{\partial\phi_1}{\partial u_1}\ \ldots\ \frac{\partial^4\phi_1}{\partial u_1^2\,\partial v_1^2}$ are substituted in equation (10′), we find, after some easy reductions,

$$\frac{\partial^4\phi_1}{\partial u^2\,\partial v^2}=\left(\frac{\frac{\partial^2\phi_1}{\partial u^2}}{\frac{\partial\phi_1}{\partial u}}+\frac{\frac{\partial^2\phi_1}{\partial u\,\partial v}}{\frac{\partial\phi_1}{\partial v}}\right)\frac{\partial^3\phi_1}{\partial u\,\partial v^2}+\left(\frac{\frac{\partial^2\phi_1}{\partial u\,dv}}{\frac{\partial\phi_1}{\partial u}}+\frac{\frac{\partial^2\phi_1}{\partial v^2}}{\frac{\partial\phi_1}{\partial v}}\right)\frac{\partial^3\phi_1}{\partial u^2\,\partial v}$$
$$-\frac{\frac{\partial^2\phi_1}{\partial u^2}\ \frac{\partial^2\phi_1}{\partial u\,\partial v}\ \frac{\partial^2\phi_1}{\partial v^2}}{\frac{\partial\phi_1}{\partial u}\ \frac{\partial\phi_1}{\partial v}}-\left(\frac{\partial^2\phi_1}{\partial u\,\partial v}\right)^2\left(\frac{\frac{\partial^2\phi_1}{\partial u^2}}{\left(\frac{\partial\phi_1}{\partial u}\right)^2}+\frac{\frac{\partial^2\phi_1}{\partial v^2}}{\left(\frac{\partial\phi_1}{\partial v}\right)^2}\right).$$

Comparing this with (10), we have the result: *Given any function $\phi(u, v)$ satisfying equation* (10), *then $\phi(U, V)$ is an integral, where U and V any functions of u and v, respectively.*

In the same way it can be shown that equation (13) takes the form

$$\int\frac{\left(\frac{\partial^2\phi_1}{\partial u\,\partial v}\right)^2}{\frac{\partial\phi_1}{\partial u}\left(\frac{\partial\phi_1}{\partial v}\right)^2}du+\int\frac{\left(\frac{\partial^2\phi_1}{\partial u\,\partial v}\right)^2}{\left(\frac{\partial\phi_1}{\partial u}\right)^2\frac{\partial\phi_1}{\partial u}}dv+2\,\frac{\frac{\partial^2\phi_1}{\partial u\,\partial v}}{\frac{\partial\phi_1}{\partial u}\ \frac{\partial\phi_1}{\partial v}}+V_1-U_1=0.$$

Hence, the functions U_1 and V_1 corresponding to the solution ϕ_1 of equation (10) are gotten from those corresponding to the solution ϕ by replacing u and v by U and V. In a similar manner it can be shown that the expressions for $x-iy$ and z corresponding to the solution ϕ_1 are gotten from the corresponding ones for the solution ϕ by the same substitution. However, since a change of parameters doesn't affect the surface, and since the same lines are parametric when u is replaced by a function of u and v by a function of v, it is evident that all functions ϕ which can be brought to the same form by such a change determine the same surface.

We shall consider now the surfaces corresponding to several evident integrals of equation (10) and several which are readily found indirectly.

A particular solution of equation (10) is given by

$$\phi = uv. \tag{16}$$

When this expression for ϕ is substituted in (13), the latter reduces to

$$U - V = 0,$$

hence

$$U = V = \alpha,$$

where α is a constant. Putting these values for ϕ, U and V in (14) and (15), we get

$$x + iy = uv, \quad x - iy = \tfrac{1}{4}\left(\frac{1}{uv} - 2\alpha \log \frac{u}{v} - \alpha^2 uv\right),$$

$$z = \tfrac{1}{2}\left(\log \frac{u}{v} + \alpha uv\right). \tag{17}$$

Eliminating u and v, we get the following equation of the surface

$$4(x^2 + y^2 + z^2) = 1 + [2z - \alpha(x + iy)]^2, \tag{18}$$

Hence, unless α is zero, the surface is imaginary. When $\alpha = 0$, equation (18) reduces to

$$x^2 + y^2 = \tfrac{1}{4},$$

that is, the surface is a right circular cylinder. Recalling the preceding results, we have that *the cylinder of revolution is the only real surface of constant mean curvature corresponding to the function $\phi = UV$, where U and V are any functions whatever of u and v respectively.*

Another evident integral of equation (10) is

$$\phi = \frac{1}{u + v}. \tag{19}$$

As in the preceding case, equation (13) for this expression of ϕ reduces to

$$U - V = 0,$$

hence,

$$U = V = \alpha.$$

When these expressions for ϕ, U and V are substituted in (14) and (15), they give

$$x + iy = \frac{1}{u + v}, \quad x - iy = \frac{4uv + 2\alpha(u - v) - \alpha^2}{u + v}, \quad z = \frac{v - u + \alpha}{u + v}. \tag{20}$$

If these values for ϕ and z are put in the expressions (7) for D and D'', we get

$$D = D'' = 0,$$

that is, the lines of length zero are asymptotic lines for the surface. Hence S is a sphere.* In fact, this is readily seen by eliminating u and v from the expressions (20). If we introduce parameters u_1 and v_1 defined by

$$u_1 = 2u - \alpha, \quad v_1 = 2v + \alpha,$$

the expressions (20) become

$$x + iy = \frac{2}{u_1 + v_1}, \quad x - iy = \frac{2u_1 v_1}{u_1 + v_1}, \quad z = \frac{v_1 - u_1}{u_1 + v_1}, \tag{20'}$$

and by elimination

$$x^2 + y^2 + z^2 = 1.$$

Hence, *for any function* ϕ *of the form* $\dfrac{1}{U+V}$ *the surface is a sphere.*

We propose now to find all integrals of equation (10) which are functions of $u + v$. If we denote by accents derivatives with respect to $u + v$, we find that equation (10) reduces to the total differential equation

$$\phi^{\mathrm{IV}} - 4\frac{\phi''\phi'''}{\phi'} + 3\frac{\phi''^3}{\phi'^2} = 0.$$

Put $y = \phi'$, $p = \phi''$, then upon substitution and reduction the above equation becomes

$$p\left[p\frac{d^2p}{dy^2} + \left(\frac{dp}{dy}\right)^2 - 4\frac{p}{y}\frac{dp}{dy} + 3\frac{p^2}{y^2}\right] = 0.$$

If this equation is satisfied by $p = 0$, we have

$$\phi = c\,(u + v),$$

where c is a constant, which, as we have seen, is the case of minimal surfaces. We exclude this case and introduce two new functions, z and r defined by

$$p = 2y^{\frac{3}{2}}z, \quad y = e^r.$$

* Bulletin of the Amer. Math. Soc., March, 1902, p. 241.

Then the above equation reduces to

$$z\frac{d^2z}{dr^2}+z\frac{dz}{dr}+\left(\frac{dz}{dr}\right)^2=0.$$

An integral of this equation is $z=$ const., say c; then, from the above we have

$$\phi''=2\phi'^{\frac{3}{2}}c,$$

whence, by two quadratures,

$$\phi=-\frac{1}{c^2(u+v)},$$

which, as we have seen, leads to the sphere.

We exclude this case now and put

$$\frac{dz}{dr}=t;$$

then the above equation becomes

$$z\frac{dt}{dz}+t+z=0,$$

which, upon integration, gives

$$t=\frac{c}{z}-\frac{z}{2}.$$

Retracing the steps in the above substitutions, we get

$$\phi=\gamma\tan\frac{u+v+\beta}{\alpha}+\delta, \tag{21}$$

where α, β, γ, δ are arbitrary constants. By a translation of the surface and a change of parameters, this expression for ϕ can be given the form

$$\phi=x+iy=\gamma\tan(u+v). \tag{21'}$$

The equation (13) reduces to

$$\frac{4(u+v)}{\gamma}+V-U=0,$$

from which we have

$$U=\frac{4u+2\delta}{\gamma},\quad V=-\frac{4v-2\delta}{\gamma},$$

where δ is a constant. From (14) and (15) we find

$$\left.\begin{aligned} z &= (u - v + \delta)\tan(u+v), \\ x - iy &= -\frac{1}{\gamma}\left[\frac{u+v}{2} - \tfrac{1}{4}\sin 2(u+v) + (u - v + \delta)^2 \tan(u+v)\right] \end{aligned}\right\}. \quad (22)$$

When a surface is referred to its lines of length zero, the directions of the lines of curvature are given by*

$$D\,du^2 - D''\,dv^2 = 0. \quad (23)$$

If the values of D and D'' for the surface given by (21′) and (22) are calculated by means of (7), we get for (23)

$$du^2 - dv^2 = 0.$$

Hence, if we put $\qquad u + v = u_1, \quad u - v = iv_1,$

u_1 and v_1 are the real parameters of the lines of curvature such that the linear element takes the form

$$ds^2 = \lambda\,(du_1^2 + dv_1^2).$$

The cartesian coordinates have the following expressions :

$$x + iy = \tan u_1, \quad x - iy = -\left[\frac{u_1}{2} - \tfrac{1}{4}\sin 2u_1 - v_1^2 \tan u_1\right], \quad z = iv_1 \tan u_1,$$

from which it follows that the surface is imaginary. Combining the above results with the theorem which we established just before the discussion of these particular integrals, we have the theorem :

Minimal surfaces and the sphere are the only real surfaces of constant mean curvature for which ϕ is a function of the sum of any function of u and any function of v.

Again we seek the integrals of equation (10) which are a function of uv. If we denote by accents the derivatives with respect to uv, equation (10) becomes

$$uv\,\phi'^2\phi^{\mathrm{IV}} + 2\phi'^2\phi''' - 3\phi'\phi''^2 + 3uv\,\phi''^3 - 4uv\,\phi'\phi''\phi''' = 0.$$

Put $\phi' = e^t$, $\log uv = r$; then this equation becomes

$$\frac{d^3t}{dr^3} - \frac{dt}{dr}\,\frac{d^2t}{dr^2} - \frac{d^2t}{dr^2} = 0.$$

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* Bianchi, Lezioni, p. 99.

An integral of this equation is given by $t =$ const.; in this case

$$\phi = c\,uv,$$

which case has been considered before. Excluding this case from what follows, we make the substitution $\dfrac{dt}{dr} = \theta$, and find that θ satisfies the equation

$$\frac{d^2\theta}{dr^2} - \theta\frac{d\theta}{dr} - \frac{d\theta}{dr} = 0.$$

This equation also is satisfied by θ const., say c; then

$$\phi' = e^t = \alpha e^{cr} = \alpha\,(uv)^c.$$

But this is reducible by a translation of the surface to the general form $\phi = UV$, and hence belongs to the first class considered. We make an exception of this case and put $\dfrac{d\theta}{dr} = p$; the equation in p is

$$\frac{dp}{d\theta} - \theta - 1 = 0,$$

and consequently,

$$p = \frac{\theta^2 + 2\theta + \alpha}{2},$$

where α is a constant. From this we get

$$dr = \frac{2d\theta}{\theta^2 + 2\theta + \alpha}.$$

Three cases arise according as α is greater, equal to or less than unity.

1°. $\alpha > 1$. In this case

$$r + \beta = \frac{2}{\sqrt{\alpha - 1}}\tan^{-1}\frac{\theta + 1}{\sqrt{\alpha - 1}},$$

where β is a constant. Retracing the steps, it is readily found that

$$\phi = \frac{2\gamma}{\sqrt{\alpha - 1}}\tan\frac{\sqrt{\alpha - 1}\,(\log uv + \beta)}{2} + \delta.$$

If we replace γ by $\dfrac{\sqrt{\alpha - 1}}{2}\gamma$, $\dfrac{\sqrt{\alpha - 1}\,(\log u + \beta/2)}{2}$ by u, $\dfrac{\sqrt{\alpha - 1}\,(\log v + \beta/2)}{2}$ by v and δ by zero, this expression for ϕ becomes the same as (21′). Hence this case is the same as the general case where ϕ is a function of $U + V$.

2°. $\alpha = 1$. Now

$$r + \beta = -\frac{2}{\theta + 1},$$

and, consequently,

$$\phi = \frac{-\gamma}{\log uv + \beta} + \delta.$$

By a suitable choice of parameters this becomes

$$\phi = \frac{1}{u + v},$$

which has the sphere for the corresponding surface.

3°. $\alpha < 1$. Then

$$r + \beta = \frac{1}{\sqrt{1-\alpha}} \log \frac{\theta + 1 - \sqrt{1-\alpha}}{\theta + 1 + \sqrt{1-\alpha}}.$$

From this it is found that ϕ takes the form

$$\phi = \frac{-\gamma}{\sqrt{1-\alpha}\,[(uv)^{\sqrt{1-\alpha}} e^{\beta\sqrt{1-\alpha}} - 1]} + \delta.$$

By a convenient choice of parameters and a translation, this can be written

$$\phi = \frac{1}{uv - 1}.$$

Equation (13) becomes

$$2 \log uv + U - V = 0,$$

and, consequently,

$$U = -2(\log u + \delta), \qquad V = 2(\log v - \delta),$$

where δ is a constant. From (14) and (15) we get

$$z = -\frac{\log \frac{u}{v}(uv + 1) + 2\delta}{2(uv - 1)},$$

$$x - iy = -\tfrac{1}{4}\left[uv - \frac{1}{uv} + 2 \log uv + (uv + 1) \log^2 \frac{u}{v} \right.$$
$$\left. - \frac{\left((uv + 1) \log \frac{u}{v} + 2\delta\right)^2}{uv - 1} \right].$$

In this case it is found that the equation of the lines of curvature is

$$\frac{du^2}{u^2} - \frac{dv^2}{v^2} = 0.$$

Hence if we put

$$\log u + \log v = u_1, \quad \log u - \log v = iv_1,$$

u_1 and v_1 are the real parameters of the lines of curvature. The cartesian coordinates have then the following expressions:

$$x + iy = \frac{1}{e^{u_1} - 1}, \quad x - iy = -\tfrac{1}{4}\left[e^{u_1} - e^{-u_1} + 2u_1 - v_1^2(e^{u_1} + 1) - \frac{[(e^{u_1} + 1)\,iv + 2\delta]^2}{e^{u_1} - 1}\right], \quad z = -\frac{iv_1(e^{u_1} + 1) + 2\delta}{2(e^{u_1} - 1)}.$$

From this we see that the surface is imaginary. Since $\phi = \log UV$ belongs to the above class and leads to the minimal surfaces, we have the following theorem:

Minimal surfaces, the sphere and cylinders of revolution are the only real surfaces of constant mean curvature corresponding to the case where ϕ is a function of the product of any function of u by any function of v.

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